Legendre and Lacroix about their respective textbooks of geometry, a rivalry that had more to do with material gain than with theoretical disputes.

After an introductory chapter the author treats the following aspects: books before the printing press; the emergence of the modern textbook; the concept of books on “Éléments des Sciences”; the period of the French Revolution; the case of Lacroix; didactic books versus the teacher’s autonomy; the relations between mathematics and culture at large. Schubring centers his exposition on the ideas and the textbooks of a few mathematicians – mostly Ramus, Arnauld, Clairaut, d’Alembert, Legendre and Lacroix – describing not only the different theoretical positions that underlie the choice and presentation of contents but also external factors such as the different uses of textbooks in the classroom and the relation of textbooks to national educational policies. As the preceding list of mathematicians reveals, Schubring’s description is somewhat limited geographically. Although he devotes some attention to the discussion of the differences between textbooks in Germany and France, the situation in Italy is only briefly mentioned and that of England completely omitted.

The text is light on scholarly apparatus, retains some of the informality of an oral presentation and is illustrated by suggestive images of the title pages of celebrated mathematical textbooks.

Schubring’s little book is explicitly (and modestly) termed in the subtitle as a set of “classroom notes”; it is not intended to provide a complete or definitive study of the history of mathematical textbooks. It is nevertheless a coherent and apt publication that should be welcomed, functioning, as it does, as an aperitif to a fascinating topic whose richness is still far from having been exhausted.

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The Rise and Development of the Theory of Series up to the Early 1820s

The theory of series prior to Cauchy … appears as a corpus of manipulative techniques lacking in rigor whose results seem to be the puzzling fruit of the mind of a magician or diviner rather than the penetrating and complex work of great mathematicians (p. vii).

The passage above is from the Preface to Giovanni Ferraro’s The Rise and Development of the Theory of Series up to the Early 1820s, the latest volume, apparently the 47th, in Springer’s venerable and respected Sources and Studies in the History of Mathematics and Physical Sciences series. Ferraro sets an ambitious objective, to try to untangle the many threads and to find some direction in the dead ends and false starts in the story of series, mostly in the 18th century. For the most part, he is successful, although some pieces do not fit as well as others. However, as strong as the technical content of the book is, it is seriously scarred by a great many editorial shortcomings including poor and inaccurate illustrations, a very weak index, imprecise quotations and spotty proofreading.

Ferraro’s main tool is a distinction made in Chapter 7 between the manipulation of quantities and the formal manipulation of symbols. By a quantity, Ferraro means to emphasize an entity’s “capability to increase or diminish,” while a quantum is “a specific determination of quantity” (p. 101). A calculation becomes a formal manipulation when the objects in the calculation are no longer required to represent quantities.

In his Preface, the author gives a clear statement of the thesis of his work, but the text seldom, if ever, refers to it. As a result, the reader must repeatedly refer back to the Preface to gauge Ferraro’s progress towards defending that thesis. This may be an artifact of the development of this book, since about half of the text, 15 of its 33 chapters, is adapted from five papers that the author published between 1998 and 2007, and some of the seams from the process of merging those papers together are still visible.
The first chapter begins with Archimedes’ quadrature of the parabola, using double *reductio ad absurdum*. This rigor became the “model of rigorous mathematical reasoning” (p. 4) for calculus and analysis to which mathematicians in the 17th and 18th centuries aspired. When Cauchy wrote that he “sought to give them [the theorems of analysis] all the rigor that one requires in geometry” (Cauchy, 1821, p. ii, as quoted on p. 347), this was the standard of rigor to which he aspired.

After Archimedes, the story moves very quickly to the 16th and 17th centuries and the work of Viète, Saint-Vincent, Bombelli, Mercator and others. Ferraro pauses regularly to give good, clear accounts of the mathematical details. For example, he spends four pages on Pietro Mengoli, perhaps best known for posing the so-called “Basel problem,” to find the sum of the reciprocals of the square integers. Mengoli also gave an early proof that the harmonic series diverges. Ferraro recounts Mengoli’s proof in modern terms, first showing that for positive integers \( n \geq 2 \),

\[
\frac{1}{n-1} + \frac{1}{n} + \frac{1}{n+1} > \frac{3}{n},
\]

then using this to show that if \( S \) were the sum of the harmonic series, then \( S > 1 + S \), clearly impossible if \( S \) is finite.

Two results of Leibniz become sub-plots that recur throughout the book. Starting with the observation that

\[
\frac{a}{b+c} = \frac{a - ac}{b^2 + bc} = \frac{a}{c} - \frac{ab}{bc + c^2},
\]

Leibniz expanded \( \frac{a}{b+c} \) in two different ways, namely

\[
\frac{a}{b} - \frac{ac}{b^2} + \frac{ac^2}{b^3} - \ldots
\]

and

\[
\frac{a}{c} - \frac{ab}{c^2} + \frac{ab^2}{c^3} - \ldots.
\]

The first of these converges when \( c < b \), while the second converges when \( c > b \). Thus, they cannot both be quantities at the same time. One of them (or both, in the case \( c = b \)) must be a formal manipulation. Yet, says Ferraro, “There is no reason to refuse one of the two different expansions preliminary. Only, *a posteriori* can one choose between two different developments according to the geometric situation” (p. 37). Incidentally, Isaac Newton and Jacob Bernoulli also made use of these expansions.

The second result of Leibniz is called the “Leibniz analogy.” It consists of the observation that the coefficients in the expansion of the binomial \( (x + y)^n \) are the same as the coefficients of the \( n \)th order differential of a product, \( d^n(xy) \). This opens the door for series in which the terms are no longer quantities but instead are other kinds of objects like differential operators.

Ferraro gives a fine account of the evolution of the objects in a series. Early series were either finite series of numbers or series inspired by geometry, like Taylor series derived from curves. Chapters 13 to 19 document the gradual evolution of series involving more formal and less geometric objects. Euler’s 1737 summation of

\[
\frac{1}{3} + \frac{1}{7} + \frac{1}{8} + \frac{1}{15} + \cdots = \sum_{m,n \geq 1} \frac{1}{m^n - 1},
\]

for example, involved a new kind of series. First, it did not arise from any kind of geometric or analytic question. It is simply a series of terms each satisfying some property, namely that they are reciprocals of numbers one less than powers of integers. Second, there is no practical way to give the \( n \)th term in this series, or, given a term, to find what the next term is. It does not have a naturally defined general term, nor is it recursive.

Over the course of the 18th century, and largely at the hands of Euler, the study of series relied more and more on formal manipulations. Mathematicians became bolder and bolder in extending properties of finite series to infinite series and in extending properties of convergent sums of quantities to sums of other kinds of objects, without regard to
We see series of complex numbers, of functions and of operators. We see generating functions, which, in a sense, capture the information that defines a recursive sequence, and are not really sums at all. And we see the theory and tools of infinite sums adapted to infinite products, continued fractions and nested radicals.

Eventually, Cauchy’s revolution of rigor sweeps away much of the elaborate structure of formal manipulations, and most of what survives is that which is rooted in the analysis of quantities. Thus Ferraro’s distinction between quantities and formal manipulation, which seems like a rather blunt tool throughout the book, turns out to be just barely sharp enough to separate the results that will survive Cauchy’s innovations unscathed from those results that will be revised or rejected.

In the end, the reader is left satisfied in some ways, dissatisfied in others. The reader sees a very good account of the story of series in the 1700s. Mathematical and technical details are explained with exceptional clarity, and this alone is enough to justify the place of this volume in such a prestigious series. It is a bit frustrating that we do not learn what happens to the “orphan” types of series, the series that are not quantities, but nonetheless survive the restoration of rigor in one way or another. Such series include generating functions, asymptotically convergent series and series of operators, among others. But history is obstreperous, and not all its threads ended with Cauchy in 1821, so the fates of these objects are rightly left to another volume, and perhaps another author.

It is the chapter on Maclaurin in which the strengths and weaknesses of this book are particularly exemplified. On one level this reviewer found Ferraro’s treatment of Maclaurin’s work on series particularly informative. He tells us that Maclaurin’s Treatise of Fluxions was “part of an extreme attempt at laying the foundations of calculus on a geometrical framework” (p. 147). Eighty years later, Cauchy showed particular respect for Maclaurin’s efforts, and until I read Ferraro’s account, I did not really understand why.

But on another level, this reviewer found the chapter on Maclaurin particularly disappointing, as all of the weaknesses in the production of Ferraro’s book are highlighted here. For example, Maclaurin’s Treatise is quoted 13 times, and 11 of those 13 are misquotes. Errors range from the simple omission of the word “always” (p. 147) to the drastic rewording of Proposition 20 and its two corollaries (pp. 149–150). Ferraro’s Figure 15 is adapted and dramatically simplified from Maclaurin’s Figure 86 (Maclaurin, 1742, Plate X, p. 226) and it is described as if the diagram accompanied Maclaurin’s Proposition 14. In fact, Proposition 14 used a far simpler diagram, Figure 50 (Maclaurin, 1742, Plate VII, p. 190), though a typographical error in Maclaurin cites Figure 46. Maclaurin’s Figure 86, reproduced as Ferraro’s Figure 15, originally accompanied Proposition 20 (Maclaurin, 1742, p. 219), although Ferraro’s account of Proposition 20 does not mention this diagram. Moreover, this Figure 15 is of only mediocre technical quality. Lines in the diagram unnecessarily obscure two of the labels on points, and a line segment that is supposed to be tangent to the curve is misaligned by about ten degrees. The other diagram in this chapter, Figure 16, is even worse, with labels on points strewn far from their corresponding points and ordinate segments extending beyond their corresponding curves and axes.

Other figures in other chapters suffer many more such problems. The “circle” in Figure 17 has its vertical diameter about ten percent longer than its horizontal one. The step function in Figure 18 intersects its curve in places it should not, and fails to reach the curve in places it should. Both Figures 12 and 13 are badly pixilated, and the “circles” in the former are clearly elliptical. Meanwhile, Figure 14 is a riot of segments and circles missing the points where they should intersect, a semicircle failing to reach its diameter, “equal” line segments that in fact differ by about 20 percent, and half a dozen labels for points that are not marked in the diagram. This last diagram is in the midst of an otherwise well-crafted and illuminating discussion of how a divergent series studied by Guido Grandi in 1703 helped to inspire Maria Agnesi to study the curve that bears her name. It is like a fine garden, marred by weeds.

Other things that the editors should have noticed and fixed include that function names “sin” and “log” are sometimes italicized, sometimes not, even on the same page, and at least once in the same expression (p. 275). There are more than the usual number of spelling errors, and the index was not very useful at all. For example, pages in the Preface were not indexed and, while there are 14 entries for “Continuity,” there are no sub-entries for “G-continuity” or “L-continuity,” two versions of continuity discussed in Chapters 18 and 23, nor is there an entry for “Uniform continuity,” although it is an important theme that recurs several times.

A reader would be justifiably dissatisfied with the editorial shortcomings described above. Modern publishing practices like CRC (i.e. Camera Ready Copy) give the author more control over the finished product, but such practices can also deny the author access to the training and experience that a good editor will have in style, copy editing, grammar and spelling. An author, particularly one writing in a second or third language, however expert he or she may be, cannot develop these skills to the same level as a good editor, and this volume suffers for that.
The Geometry of an Art: The History of the Mathematical Theory of Perspective from Alberti to Monge
{xviii+802 pp. $199.00

One of the most attractive topics in the history of mathematics is the development of single focused perspective. Elementary—but by no means trivial—geometry is involved, the opportunity arises to talk about architecture and renaissance paintings, connections can be made to a rich slice of Western culture, and books can be enlivened with reproductions of marvellous artworks. Interesting creative people from Piero della Francesca to Girard Desargues and J.H. Lambert can be brought in, mathematics seems both useful and beautiful, and the historian has an easy time sounding important. Every time this story has been told, however, some more critical historians of mathematics have wondered just how accurate it is. Specifically: how did the elaboration of principles and theorems in the theory of perspective affect the practise of draughtsmen and especially artists? Were there indeed any links at all? It is this fundamental question that Kirsti Andersen has addressed in the first full-length exploration of the topic, and what she presents is not only a remarkably extensive coverage but a careful analysis of the many different aspects of the mathematical theory of perspective in a richly satisfying historical context.

Understanding this mathematical theory can mean many things. It might be, for example, that you wish to explain this ‘perspectivist’ way of drawing realistic pictures according to a somewhat simplified theory of vision. Or you might wish to make an accurate picture of a specific object, whose shape and size is known to you. Or, you might wish to depict an entirely imaginary view in this realistic manner. For the first activity you might imagine, or even have, a genuine object, a screen (on which the picture will be drawn), an eye piece, and lots of threads or pinpricks in the screen that establish the lines joining points on the object to points on the screen and then the eye piece. The second activity requires that you carry out the first without the cumbersome threads; you have learned the theoretical basics, and perhaps the given object exists only in your mind (say, as a dodecahedron). The third activity is much more elaborate, and even if your picture is the interior of a building with some columns and some furniture it involves several activities of the second kind. If, moreover, your imaginary objects are people then some considerable thought and work is involved.

As this book shows in detail, each of these activities has its historical story. What is more, as people did them they found very quickly that there were rules that had to be followed. Once a few points were in place you could not put down the images of others at will: their places were determined by choices you had already made. What these rules were, what, to a mathematician, were the new theorems in this art, and which of them were fundamental and had the others as consequences was a matter of increasing interest. The reader of this review should now attempt to draw a plausible picture of a regular octahedron resting on a horizontal plane on one of its flat, triangular faces. This will not only acquaint you with the problems in activity two, it will raise an important question that Kirsti Andersen pursues throughout this book: what techniques are allowed in constructing such pictures?
The theory of series in the 17th and 18th centuries poses several interesting problems to historians. Most of the results derived from this time were derived using methods which would be found unacceptable today, and as a result, when one looks back to the theory of series prior to Cauchy without reconstructing internal motivations and the conceptual background, it appears as a corpus of manipulative techniques lacking in rigor whose results seem to be the puzzling fruit of the mind of a magician or diviner rather than the penetrating and complex work of great mathematicians. From the reviews: “Giovanni Ferraro’s book must be regarded as an important contribution to the history of mathematical analysis.”