THE SAME BUT DIFFERENT - NOVICE UNIVERSITY STUDENTS SOLVE A TEXTBOOK EXERCISE

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The aim of this paper is to illustrate some aspects of the transition between secondary and tertiary studies in mathematics, based on an analysis of a critical case of two students trying to solve a system of linear equations with help from their teacher. In their conversation, different aspects of the transition appear, which both can be assigned to changes in the mathematical content and differences in the way mathematics should be regarded and communicated. Students draw on a school mathematical discourse, while the teacher answers them within a scientific mathematical discourse. From a student perspective, mathematics studies at university compared to upper secondary school is neither more of the same, nor something brand new.

Keywords: transition, secondary, tertiary, school mathematics, mathematics teacher

This paper reports from a case study of the secondary-tertiary transition, which is understood as novice students’ learning of university mathematics according to their previous experiences of mathematics at upper secondary school. The study is based on an episode with two novice teacher students, working with a university textbook exercise in linear algebra with help from their teacher. The aim with the paper is to examine the transition in terms of an encounter between school mathematics and mathematics as a scientific discipline. After a short overview of some earlier research concerning the transition and students’ encounters with university mathematics, presentation of data is alternated with analyses of the two students working with the exercise with help from their teacher. The paper ends with conclusions and final comments about their transition between secondary and tertiary mathematics.

EARLIER RESEARCH ON THE SECONDARY-TERTIARY TRANSITION

Nowadays, the secondary-tertiary transition is a well-researched area. This interest may be due to the problems and difficulties that many students encounter when they begin studying mathematics at university. However, as our knowledge about the transition increases, it also becomes obvious that it is a complex issue and can be studied from various perspectives that explicitly or implicitly point at different crucial aspects of students’ learning of mathematics in a new learning environment (de Abreu, Bishop & Presmeg, 2002; Gueudet, 2008; de Guzmán, Hodgson, Robert & Villani, 1998). The transition has been closely connected with the transition from elementary to advanced mathematics as well as from elementary to advanced mathematical thinking (Artigue, Batanero & Kent, 2007), specifically focusing students’ encounters with mathematical abstraction (Nardi, 2000). Lithner (2003) offers a more specific study of university students’ work with textbook exercises and developed different categories of students’ reasoning according to what extent the
reasoning is based on more superficial character of the exercise or relies on more intrinsic mathematical features in the task. For some researchers, the transition is understood as a rite of passage (Clark & Vovric, 2008). Students leave a familiar environment to begin mathematics studies at university under new terms and conditions. In relation to the transition the new institutional context where the studying takes place is crucial, but also students’ actions and statements in relation to new demands on learning mathematics. This complex relation between institution and subject has been studied within the anthropological theory of didactics where the transition can be regarded as a movement between different praxeologies (Alveres, Dias, Artigue, Jahn & Campos, 2010).

SCHOOL MATHEMATICS AND MATHEMATICS AS A SCIENTIFIC SUBJECT

As an attempt to capture the tensions between secondary and tertiary mathematics, I have chosen to use the notions of school mathematics and mathematics as a scientific discipline. These can be regarded as two distinct communities of discourses (Sfard, 2007) to describe differences between treatment of the mathematical content and the learning situation at a social institution. According to Sfard (2007) a mathematical discourse consists of mathematical words, visual mediators, routines and narratives. A mathematical discourse is reflected by its use of words and visual mediators that are used by its interlocutors. Narratives are any written or spoken text that is used within the discourse and routines refer patterns in the interlocutors’ actions respectively. It is plausible to suppose that the mathematical discourse differs between secondary and tertiary mathematics education. I have chosen to give an account for these differences in terms of school mathematics and mathematics as a scientific discipline.

Even though the notion of “school mathematics” is frequently used, it seems to lack a scientifically grounded definition, but is used in a more common sense and everyday manner. In this paper, I will associate school mathematics with teaching methods that are commonly used in Swedish secondary school mathematics classrooms. Mathematics lessons starts with the teacher giving a short demonstration of some standard examples. The rest of the lesson, the pupils work individually with textbook exercises that are similar to the ones that have been demonstrated at the black board. A main focus is to learn what to do, i.e. an emphasis on algorithms and procedures. Mathematics as a scientific discipline concerns mathematics as it is considered within an academic and scientific discourse, for example university courses in mathematics (Robert & Schwarzenberger, 1991). The use of exact definitions, theorems and proofs is significant, which is often communicated through generic examples.

Thus, a dichotomy between mathematics studies at secondary and tertiary level can be identified, which from a students’ perspective can be regarded as a process of enculturation into the university mathematical discourse, based on their previous knowledge and understanding. I define this way to treat and work with mathematics as a functional understanding (Stadler, 2009), which shall be understood as students’
attempts to tell new narratives and undertake new routines within a mathematical discourse that is more in accordance with mathematics as a scientific discipline, even though the students are not yet in all respects ready.

METHOD

The data, presented in this paper, is selected as a critical case from a more comprehensive study about the transition (Stadler, 2009) where five teacher students in mathematics were studied during their first semester of mathematics studies in Calculus 1 and Algebra during a time period of ten weeks, and Calculus 2 and Linear Algebra during another ten weeks period. The students were frequently interviewed and observed during lectures, tutorials and their individual work with textbook exercises. The observed episode took place during the first week of the linear algebra course and was chosen for this paper because it constitutes a critical case of the secondary-tertiary transition, understood as students’ learning of mathematics in a new setting with regard to their previous experiences. The notion of a critical case comes from Flyvbjerg (2006) and can be defined as “having strategic importance in relation to the general problem” (p. 229). The episode is orbiting a textbook exercise that requires both routines and a more advanced mathematical thinking and an interaction with the mathematics teacher at university. Thus, in the situations there are many possibilities for an interface between previous experiences on school mathematics and a new encounter with mathematics as a scientific subject to arise that makes the situation a critical case of the transition. The episode was audio recorded and transcribed in full. The transcriptions were analysed by methods inspired by grounded theory (Charmaz, 2006) by using constant comparisons, memos, sorting and categorisation. However, instead of generating own categories, the analysis was focused on empirical instances of discursive elements of school mathematics and mathematics as a scientific discipline.

RESULTS AND ANALYSIS

Linear algebra has been regarded as a mathematical domain that seems to be cognitively difficult for students to learn (Dorier & Sierpinska, 2001). Vector spaces and their origins is one area that can be particularly problematic for novice university students. However, in the first course in linear algebra at the university where the study took place, the course mainly treats systems of linear equations, matrices, determinants, dot and cross products, vectors, changes of basis, eigenvalues and linear transformations in two and three dimensions. This means that most of the concepts that are treated in the course can be geometrically illustrated. It also emphasises important aspects of the conversion between algebraic and geometric representations of mathematical objects and content.

It is against this background that the task, which Jenny and Ellen are working with, should be regarded. The exercise is number 14 in the first chapter of their textbook in linear algebra. The exercise is formulated as
Solve the system of equations for all values of the constants \( a \) and \( b \).

The students attempt to solve the task by successive elimination. The third equation is reduced to:

\[
\left(-\frac{1+a}{2}\right)z = -9 + b
\]

To proceed, the students would have to analyse different cases depending on the values of the constants. In particular, whether the constant \( a \) equals minus one or not will generate different types of solutions of the system of linear equations. However, Jenny and Ellen get stuck and ask the teacher for help.

Ellen: We want to find the values of these constants. We understand that if \( a \) equals minus one, then the brackets equals zero, and then \( b \) will equal nine.

Teacher: Yes, well, yes. Well, I think we should start with the basic problem here, because you expressed things slightly incorrect. You said that you should determine these constants, \( a \) and \( b \). But that is not what you should do. You should, for all values on \( a \) and \( b \), find the solutions of the system of equations.

Ellen: Okay.

Teacher: Right. So do you get the basic question here? In other words, if I say \( a \) equals 17 and \( b \) equals minus five, what is the solution of the system?

Ellen: Okay.

Teacher: Then, you should be able to give me an answer, whatever values I give \( a \) and \( b \).

The task that Ellen and Jenny are working with has a specific difficulty. It requires the students to simultaneously handle variables, constants and parameters. These unknowns have different meanings and functions for the equations and consequently for the solution of the exercises and should be treated in different ways. The teacher’s emphasis on finding solutions of the system for all values on \( a \) and \( b \) can be interpreted as an attempt to highlight this distinction. From a school mathematical context, students seem used to that unknowns, represented by a letter, should be decided or determined by finding their values. Accordingly, Ellen’s initial statement that they want to find the values of the constants can be interpreted as a misunderstanding of the aim with the exercise, but also as an everyday and an operational way to describe how they have worked with the task.

Instead of giving the students a more direct answer to their questions, the teacher wants to give a more comprehensive explanation, which can be interpreted as making an effort to highlight the generic character in the exercise concerning role and function of the constants. Ellen and Jenny ask for a routine of how they should proceed to determine the values of \( a \) and \( b \). The teacher gives them an account of the role of the constants in the system of linear equations. These different approaches can be interpreted as an expression of the transition, where the teacher focuses the general
Further work with the task results in the expression: \[ z = \frac{18 - 2b}{1 + a} \]

Jenny: If you divide with \( a \), it should not be…

Teacher: Yes, exactly! Go on!

Jenny: Minus one.

Teacher: Yes, that’s right. If we want an expression for \( z \), we must divide by one plus \( a \) on both sides. And then, we have to be careful when, as you said, Jenny…

Jenny: …when \( a \) equals minus one.

Teacher: Yes. Exactly. So you must start looking at… But, then you must not say that \( a \) is or must not equal minus one. You should examine both cases. If \( a \) does not equal minus one, then one plus \( a \) does not equal zero and then there is no problem with division and we can get an expression for \( z \). But you must not stop there, you must also examine the case when \( a \) equals minus one.

That division by zero is not an acceptable mathematical operation seems familiar to Jenny, but it is a considerable abstraction from knowing this for natural numbers to apply it on a rational algebraic expression. However, Jenny seems to have taken this step. One interpretation of Jenny’s statements and discussion with the teacher about this is that she rather confirms this knowledge in relation to a very local mathematical context, namely the algebraic expression at hand. The teacher may or may not reflect on this, but his focus is clear; to involve the overall and more embracing ideas of solving systems of linear equations and how a division with zero should be interpreted in that context.

The students and the teacher re-write the system of equations for the case when \( a = -1 \) and solve it.

Teacher: Okay, and now \( a \) equals minus one. Let me write a little bit here. That equals minus two \( z \) there [refers to the second reduced equation]. And in the last equation, well, if \( a \) equals minus one, the left hand side will equal zero. And then we have: zero equals minus nine plus \( b \) \([0 = -9 + b]\). So, re-write the system of equations once again for the case when \( a \) equals minus one, and use that as a point of departure for thinking. Which situation do we have now according to the last equation?

Jenny: That one?

Teacher: Yes. What conclusions can you draw?

Jenny: That if \( a \) equals minus one, then \( b \) equals nine.
Teacher: Well, we agreed that $a$ and $b$ should be free. You must not say that $b$ must equal nine. If I ask you the question, if $b$ equals ten?

Ellen: Then this is not valid. Then we don’t have any solutions.

Teacher: No. Exactly. Thus, if $b$ equals ten, then the last equation says that zero equals one. And the conclusion must be that no solution exists for $b$ equals ten.

Ellen: Mm.

Jenny: But how do we write this in our solution? Solutions only exist if $b$ equals nine or what?

Teacher: Yes, but we shall push things one step further and analyse which solutions we get, because we have to organize things in different cases, partial solutions.

Ellen: Oh gosh!

Teacher: But didn’t you follow my example?

Ellen: Yes!

The students discuss with the teacher about how they should move on. They consider the case when $b = 9$ and the system of equations has infinite many solutions.

Teacher: So, an interesting case is when $b$ equals nine, because then we have $x$ plus $y$ plus $z$ equals five. We have minus two $y$ minus two $z$ equals minus two. And finally, the last condition, zero equals zero. So, what about the last equation?

Ellen: We don’t need to care about that one.

Teacher: You have to have a parameter there. You’ll get infinite many solutions in that case. So, if I say that $z$ equals four, then $y$ becomes something and we put it up there and $x$ becomes something. And if $z$ equals seven, well, then we’ll get a $y$-value and then an $x$-value.

Regarding the last equation, $0 = -9 + b$, the teacher asks the students to analyse and draw conclusions. Jenny’s and Ellen’s various responses to the teacher can be interpreted as that they are working in a school mathematical context with familiar mathematical routines as equations and writing down complete solutions. Once again, the teacher emphasises that the constants $a$ and $b$ do not have any specific values but should be regarded as “free”. As mathematical words within a discourse, these constants get a new meaning. Even though Jenny gets a remark from the teacher not to fix the values $a$ and $b$, her statement “if $a$ equals minus one, then $b$ equals nine” could be interpreted as a first sign of understanding the new way of dealing with different cases and treating unknown constants, even if it is just a conclusion of what she thinks the teacher wants her to say. Her question about how the solution should be written indicates a focus on routines that differs from the teacher’s but also serves
as an attempt to take part in a new way of using narratives for and solving a mathematical textbook exercise.

At the same time, Ellen manages to draw a correct conclusion about what happens if the last equation ends up in a contradiction or both sides equal zero, but seems to be put under stress when the teacher wants to push the discussion a little bit further. The teacher constantly continues to push things towards a more scientific mathematical context, where the solution of equations is not an aim in itself but rather a tool for drawing more general mathematical conclusions.

The students and the teacher discuss the case when $b$ does not equal nine. The last equation $0 = -9 + b$ becomes a contradiction, and in this case the system will have no solution. Then the case when $a$ does not equal minus one is discussed.

Ellen: How do we do with this case when $a$ does not equal minus one? Shouldn’t we do some kind of “partial solutions”?

Teacher: In general, it depends, but in this case, I can tell, it won’t be needed. Because there you have the equation when $a$ doesn’t equal minus one. Then you can divide by one plus $a$ on both sides to get $z$ and then there won’t be any problems.

Ellen: Sorry, what did you say about… What were we supposed to do here, you said? If… if $a$ does not equal minus one?

Jenny: Then we can get $z$ and put it in the…

Ellen: But then we can have a $z$ that does equal anything and contains both $a$ and $b$?

Jenny: $a$ and $b$…

Teacher: Yes, and that is not an unnatural thing in any way. If you think about, well, of course the whole system depends on the parameters $a$ and $b$. So it is self-evident that the solutions do so too. Or, well, it isn’t self-evident but it isn’t unnatural either, so to say.

Ellen: Mm…

The teacher leaves the students, who try to finish the solution of the exercise.

Ellen: But, should we do something more with this case?

Jenny: I don’t think so.

Ellen: We shouldn’t change it with a $t$ or something like that?

At a first glance, Ellen’s final comments can be interpreted as self-ironic, directed towards their own mathematical ignorance and inability. However, they might also be interpreted as a fist attempt to use narratives within a more scientific mathematical discourse. Ellen seems to have grasped that just solving a system of linear equations may sometimes not be enough, that several cases and options may occur during the
work. The routines for this kind of exercise might have changed. Maybe Ellen has not has fully grasped how to deal with different cases, but using the word and trying to fit it in a suitable statement can be regarded as an initial step towards taking part in a more scientifically oriented mathematical discourse.

A main feature in the data is the discrepancy between what the students regard as their main problem while working with the exercise and the help and explanations that are offered by the teacher. The students seem to work and understand things according to a mathematical discourse that may be described as school mathematics that is based on their previous experiences from upper secondary school.

One difficulty for the students is to deal with three different kinds of unknowns; variables, constants and parameters. The students seem to be used to that an expression with unknowns should be solved or treated as an equation. The unknown values should be found, preferably as numerical values. To handle the system of equations by finding $x, y$ and $z$ and simultaneously taking all values on $a$ and $b$ into consideration can be a difficult thing to do. A simpler way to think about the solution is to solve the system for specific values of $a$ and $b$. However, one should not conclude that this is due to students’ misconception. Rather, it can be regarded as a pragmatic way to handle a complex situation. What seems to be an immature way to think and act is simply the way that they manage to think and act, because that is the way that they are acquainted and familiar with. This can be regarded as an empirical instance of functional understanding. The students draw on pre-knowledge and make their best efforts to blend it into the new discourse of university mathematics.

**SUMMARY AND CONCLUDING REMARKS**

From interviews with students about their expectations on their upcoming university studies in mathematics, students describe the major differences in terms of quantitative change (Stadler, 2009). Mathematics studies at tertiary level are more time consuming, lectures are longer and take place more often, one has to solve more exercises and do a lot more homework. However, the core of the transition seems to be constituted of minor qualitative changes in the teaching and learning of mathematics, which can be understood as fine-tuned changes in the mathematical discourse. These differences appear in the teacher’s way of explaining tasks and the more comprehensive way to talk about mathematics. Instead of answering the students’ questions of how they should do, the teacher explains the main idea and how they should think. The teacher regards the exercise at hand as an example of a more generic mathematical entity, while the students are working in different modes of solving equations. This can be regarded as a clash between school mathematics and mathematics as a scientific discipline as two different communities of discourses.

Changes are also built in the mathematical exercise that the students are working with. On the surface, the exercise seems standardized and ordinary, but demands new way of mixing a standardized algorithmic solution with an extensive analysis of different cases and a new way of talking about the task. The switching between
different kinds of unknowns also makes the exercise complex. The students encounter situations that they interpret according to previous experiences and uses well-known routines, when these situations should be regarded and treated in a partly new way. One could argue that other more capable students would have solved the exercise in a different manner. That puts an individual perspective on the transition, referring to students’ pre-knowledge and ability to grasp the crucial aspects of the task. On the other hand, the episode offers an illustration of how these students are constantly trying to adapt their reasoning in accordance with how they understand the situation at hand. It is also shown that the transition is not just about changes in the mathematical content that the students should learn, nor is it a question of poor pre-knowledge, but a delicate interplay between these and other aspects of change.

It can be concluded that the transition from secondary to tertiary mathematics puts new demands on students’ use of mathematical words, their communication with narratives and ways of applying routines. To cope with the new learning environment that the university constitutes and the new mathematical content that they encounter there, students both have to learn a partly new kind of mathematics but also learn how to learn mathematics. This involves individual, social and institutional dimensions of the transition. From an individual perspective, variations in students’ prior experiences and knowledge have an impact on their university studies in mathematics. On the other hand, what kind of exercises the students are working with and how the teacher explains them for the students can also be a source of transitional problems for the students that can be regarded as social and institutional dimensions of the transition.

At university, students have to focus on a more comprehensive mathematical learning object. One strategy for the students is to apply a functional understanding, trying to adapt to the actual situation and to participate in the discussion with the teacher as well as possible. Instead of giving the students explanations of “what they should do”, the teacher focuses “how things are” and treats the exercise as a generic example. The teacher mediates the new expectations and demands. Students have to re-evaluate the teacher’s explanations and they also have to learn to use other sources for explanations, for example the textbook and discussions with peers, but also to draw own conclusions from more general characteristics and statements. Thus, the transition involves small qualitative and often implicit changes of the mathematical content and the setting for teaching and learning, which interact and mutually affect each other, as a shift from one community of mathematical discourse to another. Studies of mathematics at tertiary level is not something completely new, compared to students’ previous experiences but neither totally familiar.

REFERENCES


Some students choose to continue their further education at a university or polytechnic where they can study academic subjects. Other students choose to go to a college where they can study more practical subjects like art or engineering. Translate the following words into English. Students studying for first degrees are known as "undergraduates". New undergraduates in some universities are called "freshers". They learn a new way of studying which is different from that of school. They have lectures, there are regular seminars, at which one of the students reads a paper he or she has written. The paper is then discussed by the tutor and the rest of the group. 56. Same word, different meaning. In each pair of sentences below, the missing word is the same but the meaning is different. What are the missing words? 1. TRANS FATTY ACIDS A recent editorial in British Medical Journal (BMJ), written by researchers from the University of Oxford, has called Ditching that Saintly Image (IELTS). READING PASSAGE 3 You should spend about 20 minutes on questions 27 â€“ 40, which are based on Reading Passage 3. Ditching that S